

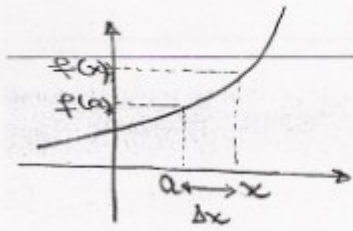
INŽENJERSKA MATEMATIKA 1

TUTORIAL 8

DIFERENCIJALNI RAČUN I UN REALNIH FUNKCIJA JEDNE REALNE
PROMJENJIVE
ZADACI + TEORIJA

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→ TUTORIJAL 8 ←

Diferencijalni račun realnihfunkcija jedne realne promjenljive

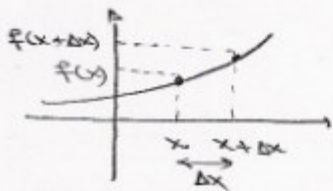
$\Delta x \rightarrow$ priraštaj argumenta u tački a

$f(a+\Delta x) - f(a) \rightarrow$ priraštaj f-je u tački a

Izvod f-je u tački a naziva se konačna
granična vrijednost

$$\lim_{\Delta x \rightarrow 0} \frac{f(a+\Delta x) - f(a)}{\Delta x}$$

i označava se $f'(a)$.



Izvod f-je f u tački a se može još pisati kao:

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ Izvod f-je u bilo kojoj tački.

Jednostrani izvodi: u tački a :

$$f'_-(a) = \lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} \quad \leftarrow \text{lijevi}$$

$$f'_+(a) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} \quad \leftarrow \text{desni}$$

→ f-ja ima izvod u tački a ako vrijedi

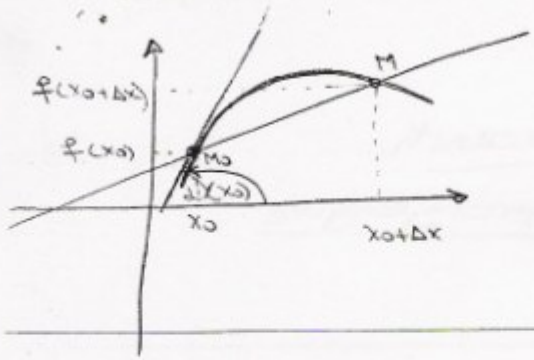
$$f'_-(a) = f'_+(a) = f'(a).$$

Beskonačan izvod

ako vrijedi

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = +\infty \text{ (ili } -\infty)$$

Geometrijsko značenje izvoda:



$$\lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \text{tg } \alpha(x_0)$$

tj. $f'(x_0) = \text{tg } \alpha(x_0)$

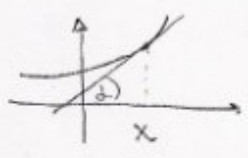
$f'(x_0)$ je koef. smjera tangente krive $f(x)$ povučene u tački M_0 .

Jednačina tangente

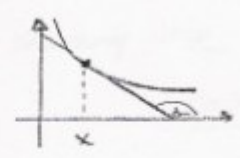
$$y - f(x_0) = f'(x_0) \cdot (x - x_0)$$

Jednačina normale

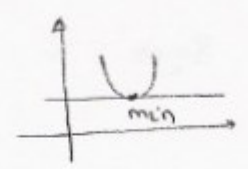
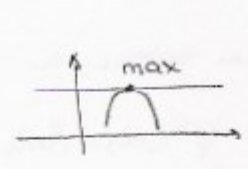
$$y - f(x_0) = -\frac{1}{f'(x_0)} (x - x_0)$$



$f'(x) > 0 \Rightarrow M \uparrow$

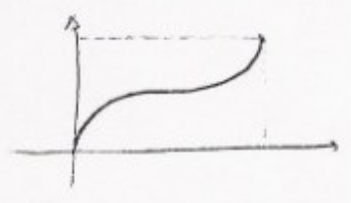


$f'(x) < 0 \Rightarrow M \downarrow$



$f'(x) = 0 \Rightarrow$ mli ekstrem f.c. je

Mehaničko značenje izvoda - definiranje brzine kod nejednolikog kretanja materijalne tačke.



trenutna brzina

$$v = \frac{ds}{dt} = \lim_{\Delta t \rightarrow 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t}$$

$$a = \frac{dv}{dt}$$

Diferencijal f.c. je:

$$df(x) = f'(x_0) \cdot \Delta x$$

Pravila diferenciranja:

1. $(f \pm g)'(x) = f'(x) \pm g'(x)$
2. $(f \cdot g)'(x) = f' \cdot g + f \cdot g'$
3. $\left(\frac{f}{g}\right)'(x) = \frac{f' \cdot g - f \cdot g'}{g^2}, g \neq 0$

Izvod složene f.c. je:

$$f(g(x))' = f'(g(x)) \cdot g'(x)$$

Izvod konstante je nula, tj. 0

Tablica izvoda:

Izraz	Izvod
x^d	$d \cdot x^{d-1}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\operatorname{tg} x$	$\frac{1}{\cos^2 x}$
$\operatorname{ctg} x$	$-\frac{1}{\sin^2 x}$
a^x	$a^x \ln a \Rightarrow (e^x)' = e^x$

$\log_b x$	$\frac{1}{x \ln b} \Rightarrow (\ln x)' = \frac{1}{x}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos x$	$-\frac{1}{\sqrt{1-x^2}}$
$\operatorname{arctg} x$	$\frac{1}{1+x^2}$
$\operatorname{arccot} x$	$-\frac{1}{1+x^2}$

$\operatorname{sh} x$	$\operatorname{ch} x$
$\operatorname{th} x$	$\frac{1}{\operatorname{ch}^2 x}$
$\operatorname{er} x$	$\operatorname{sh} x$
$\operatorname{cter} x$	$-\frac{1}{\operatorname{sh}^2 x}$

F.M.

25b.1* Polazeći od definicije izvoda sije naći izvod sije $y = x^2$ (za bilo koji x).

$$\boxed{f'(x)} = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} =$$

$$= \lim_{\Delta x \rightarrow 0} \frac{x^2 + 2x\Delta x + \Delta x^2 - x^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} (2x + \Delta x) = \boxed{2x}, \quad \forall x$$

F.M.

25b.3. Neka se tačka kreće po pravoj sa zakonom puta $s = 3t^3 - 2t^2 + 4$, gdje je s pređeni put mjeren u cm.

U momentu t (t je vrijeme mjeren u sekundama).

- naći srednju brzinu na razmaku (t_1, t_2) za $t_1 = 1$ i $t_2 = 1.1$.
- naći brzinu promjene date sije za $t = 1$.

$$1. \quad v_{\text{sr}} = \frac{\Delta s}{\Delta t} = \frac{s(t+\Delta t) - s(t)}{\Delta t} = \frac{3(t+\Delta t)^3 - 2(t+\Delta t)^2 + 4 - (3t^3 - 2t^2 + 4)}{\Delta t} =$$

$$= \frac{3t^3 + 9t^2\Delta t + 9t\Delta t^2 + 3\Delta t^3 - 2t^2 - 4t\Delta t - 2\Delta t^2 + 4 - 3t^3 + 2t^2 - 4}{\Delta t} =$$

$$= 9t^2 + 9t\Delta t + 3\Delta t^2 - 4t - 2\Delta t, \quad t=1, \Delta t=0.1 \Rightarrow \boxed{v_{\text{sr}} = 5.73 \frac{\text{cm}}{\text{s}}}$$

$$2. \quad \boxed{v} = \frac{ds}{dt} = 9t^2 - 4t = 9 - 4 = 5 \text{ cm/s}$$

II na zia

$$\overbrace{s(t_1)} \quad \overbrace{s(t_2)}$$

$$\Delta s = s(t_2) - s(t_1)$$

$$\boxed{N_{\text{sr}}} = \frac{s(t_2) - s(t_1)}{t_2 - t_1} = \frac{3t_2^3 - 2t_2^2 + 1 - 3t_1^3 + 2t_1^2 - 1}{t_2 - t_1} =$$

$$= \frac{3(t_2^3 - t_1^3) - 2(t_2^2 - t_1^2)}{t_2 - t_1} = 3(t_2^2 + t_1t_2 + t_1^2) - 2(t_2 + t_1) =$$

$$= 3(1,21 + 1,1 + 1) - 2(1,1 + 1) = 9,93 - 4,2 = \boxed{5,73}$$

F.M.

258) Odrediti vodoravne sjednice f-je:

$$a) y = \frac{x \sin x + \cos x}{x \cos x - \sin x}$$

$$\boxed{y'} = \frac{(x \sin x + \cos x)' \cdot (x \cos x - \sin x) - (x \sin x + \cos x) \cdot (x \cos x - \sin x)'}{(x \cos x - \sin x)^2} =$$

$$= \frac{(x \cos x + x \cdot \cos x - \sin x) \cdot (x \cos x - \sin x) - (x \sin x + \cos x) \cdot (\cos x - x \sin x - \cos x)}{(x \cos x - \sin x)^2}$$

$$= \frac{x^2 \cos^2 x - x \sin x \cos x + x^2 \sin^2 x + x \sin x \cos x}{(x \cos x - \sin x)^2} =$$

$$= \frac{x^2 (\cos^2 x + \sin^2 x)}{(x \cos x - \sin x)^2} = \boxed{\frac{x^2}{(x \cos x - \sin x)^2}}$$

$$b) y = \frac{2x}{1-x^2}$$

$$\boxed{y'} = \frac{(2x)'(1-x^2) - (2x)(1-x^2)'}{(1-x^2)^2} = \frac{2(1-x^2) - 2x(0-2x)}{(1-x^2)^2} =$$

$$= \frac{2 - 2x^2 + 4x^2}{(1-x^2)^2} = \frac{2 + 2x^2}{(1-x^2)^2} = \boxed{\frac{2(1+x^2)}{(1-x^2)^2}}$$

262) Naci izvode složenih f-je

$$\text{pravilo: } \boxed{f(g(x))' = f'(g(x)) \cdot g'(x)}$$

$$a) y = \underbrace{\sin(\cos^2 x)}_{f_1} \cdot \underbrace{\cos(\sin^2 x)}_{f_2}$$

$$y' = f_1' \cdot f_2 + f_1 \cdot f_2'$$

$$f_1'(x) = [\sin(\cos^2 x)]' = \cos(\cos^2 x) \cdot (\cos^2 x)' = \cos(\cos^2 x) \cdot 2 \cos x \cdot (-\sin x) \\ = -2 \sin x \cos x \cos(\cos^2 x) = -\sin 2x \cdot \cos(\cos^2 x)$$

$$f_2'(x) = [\cos(\sin^2 x)]' = -\sin(\sin^2 x) \cdot (\sin^2 x)' = -\sin(\sin^2 x) \cdot 2 \sin x \cdot \cos x = \\ = -\sin 2x \cdot \sin(\sin^2 x)$$

$$\boxed{y'} = -\sin 2x \cdot \cos(\cos^2 x) \cdot \cos(\sin^2 x) - \sin(\cos^2 x) \cdot \sin 2x \cdot \sin(\sin^2 x) = \\ = -\sin 2x [\cos(\cos^2 x) \cos(\sin^2 x) + \sin(\cos^2 x) \sin(\sin^2 x)] = \\ = -\sin 2x \cdot \cos(\cos^2 x - \sin^2 x) = \boxed{-\sin 2x \cdot \cos(\cos 2x)}$$

$$b) y = x^{(a)} + a^{(x^a)} + a^{(a^x)}$$

$$[a^{f(x)}]' = a^{f(x)} \ln a \cdot f'(x)$$

$$(x^a)' = a \cdot x^{a-1}, \quad (a^x)' = a^x \ln a$$

$$\boxed{y' = a^a \cdot x^{a-1} + (a^x) \cdot \ln a \cdot a \cdot x^{a-1} + a^{(a^x)} \cdot \ln a \cdot a^x \ln a}$$

263.b)* Dokazati da postoji (jednolična) f-je $y=y(x)$ zadane susednih jednačinam i zatim odrediti y'_x .

$$y + \ln y = x.$$

Dana jednakost definiše jednoličnu f-ju $x=x(y)$, čiji je izvod $x'_y = 1 + \frac{1}{y}$ ($=y \neq 0$). Ako je $x=x(y)$ monotona f-je \Rightarrow tada postoji jedinstvena inverzna f-je $y=y(x)$.

Ispitivanje monotonosti po def.:

$$x(y_1) - x(y_2) = (y_1 - y_2) + \ln \frac{y_1}{y_2} > 0 \Leftrightarrow y_1 > y_2 (> 0) \Rightarrow M \uparrow$$

Izvod inverzne f-je

$$\boxed{y'_x = \frac{1}{x'_y}} = \frac{1}{1 + \frac{1}{y}} = \frac{y}{y+1}$$

F.M.

Zad. 6b) Odrediti izvod zveštice i funkcije:

$$1^{\circ} y = \sqrt{\frac{1}{x}}$$

pravilo:

$$\boxed{[u(x)^{v(x)}]' = v \cdot u^{v-1} \cdot u' + u^v \cdot \ln u \cdot v'}$$
 Logaritamski izvod $(\ln y)'_x = \frac{y'}{y}$

$$y' = \left[\left(x^{-1} \right)^{\frac{1}{2}} \right]' = \frac{1}{x} \cdot \left(\frac{1}{x} \right)^{\frac{1}{2}-1} \cdot (-1) \cdot x^{-2} + \left(\frac{1}{x} \right)^{\frac{1}{2}} \cdot \ln \frac{1}{x} \cdot (-1) x^{-2} =$$

$$= - \left(\frac{1}{x} \right)^{\frac{1}{2}} \frac{1}{x^2} - \frac{1}{x^2} \left(\frac{1}{x} \right)^{\frac{1}{2}} \cdot \ln \frac{1}{x} =$$

$$= \left(\frac{1}{x} \right)^{\frac{1}{2}} \left(- \frac{1}{x^2} - \frac{1}{x^2} \ln \frac{1}{x} \right) = \left(\frac{1}{x} \right)^{\frac{1}{2}} \frac{-1 - \ln \frac{1}{x}}{x^2} =$$

$$= \boxed{\left(\frac{1}{x} \right)^{\frac{1}{2}} \frac{\ln x - 1}{x^2}}$$

$$2^{\circ} y = (\sin x)^{\ln x}$$

$$u = \sin x, v = \ln x$$

$$\boxed{y'} = \ln x \cdot (\sin x)^{\ln x - 1} \cdot \cos x + (\sin x)^{\ln x} \cdot \ln \sin x \cdot \frac{1}{x} =$$

$$= (\sin x)^{\ln x} \left[\ln x \cdot \frac{\cos x}{\sin x} + \frac{\ln \sin x}{x} \right] = \boxed{(\sin x)^{\ln x} \left(\frac{\ln \sin x}{x} + \ln x \cdot \cot x \right)}$$

$$3^{\circ} y = \sin(x^{\ln x})$$

$$y' = (x^{\ln x})' = \ln x \cdot x^{\ln x - 1} \cdot 1 + x^{\ln x} \cdot \ln x \cdot \frac{1}{x} = x^{\ln x} \left[\frac{\ln x}{x} + \frac{\ln x}{x} \right] = 2 \frac{\ln x}{x} \cdot x^{\ln x}$$

$$y' = \left[\sin(x^{\ln x}) \right]' = \cos(x^{\ln x}) \cdot (x^{\ln x})' = \boxed{\cos(x^{\ln x}) \cdot 2 \frac{\ln x}{x} \cdot x^{\ln x}}$$

F.M.

270)* Za koje vrijednosti parametra d funkcija:

$$f(x) = \begin{cases} x^d \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

a) neprekidna u tački $x=0$?~~b) ima izvod u tački $x=0$?~~c) ima neprekidan izvod u tački $x=0$?

$$\Rightarrow 0 \leq |x^d \sin \frac{1}{x}| \leq |x^d| \leq |x|^d$$

 $\lim_{x \rightarrow 0} |x|^d = 0$ za $d > 0 \Rightarrow$ na osnovu teoreme o uključivanju

imamo da je

$$\lim_{x \rightarrow 0} |x^d \sin \frac{1}{x}| = 0, \quad d > 0 \quad (*)$$

 \Rightarrow funkcija je neprekidna u $x=0$ za $\boxed{d > 0}$.

 \Rightarrow

$$f'(0) = \lim_{x \rightarrow 0} \frac{f(x+0) - f(0)}{x-0} = \lim_{x \rightarrow 0} \frac{x^d \sin \frac{1}{x}}{x} = \lim_{x \rightarrow 0} x^{d-1} \sin \frac{1}{x} = 0 \quad \text{za}$$

na osnovu

$$d-1 > 0$$

$$\boxed{d > 1}$$

$$\Rightarrow f'(x) = d \cdot x^{d-1} \cdot \sin \frac{1}{x} + x^d \cdot \cos \frac{1}{x} \cdot \left(-\frac{1}{x^2}\right) =$$

$$= d \cdot x^{d-1} \sin \frac{1}{x} - x^{d-2} \cos \frac{1}{x} \quad (x \neq 0)$$

 Na osnovu (*) je $f'(0) = 0$ za $d-2 > 0$

$$\boxed{d > 2}$$

2069) Načítá po definiciji $f'(1)$ ako je $f(x) = \sqrt{x}$.

$$\begin{aligned} \boxed{f'(1)} &= \lim_{\Delta x \rightarrow 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\sqrt{1+\Delta x} - 1}{\Delta x} \cdot \frac{\sqrt{1+\Delta x} + 1}{\sqrt{1+\Delta x} + 1} \\ &= \lim_{\Delta x \rightarrow 0} \frac{\cancel{1+\Delta x} - 1}{\cancel{\Delta x}(\sqrt{1+\Delta x} + 1)} = \boxed{\frac{1}{2}} \end{aligned}$$

2072)* Načítá $f'(x)$ za $f(x) = \cos(2x+1)$: a) u radijanech, b) x u stepenima. ~~Načítá $\frac{d^2}{dx^2}$~~

$$a) f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\cos(2x+2\Delta x+1) - \cos(2x+1)}{\Delta x}$$

$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

$$\boxed{f'(x)} = \lim_{\Delta x \rightarrow 0} - \frac{2 \sin \frac{4x+2\Delta x+2}{2} \sin \frac{2\Delta x}{2}}{\Delta x} =$$

$$= \lim_{\Delta x \rightarrow 0} - \frac{2 \sin(2x+\Delta x+1) \sin(\Delta x)}{\Delta x} =$$

$$= \lim_{\Delta x \rightarrow 0} (-2) \sin(2x+\Delta x+1) \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x} =$$

$$= \boxed{-2 \sin(2x+1)}$$

$$b) \lim_{\Delta x \rightarrow 0} \frac{\sin \Delta x}{\Delta x} = \frac{\pi}{180} \Rightarrow$$

$$\boxed{f'(x) = -\frac{\pi}{90} \sin(2x+1)}$$

2085)* Naci: $f'_-(0)$ i $f'_+(0)$. Postoji li $f'(0)$ za f.c.je

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$f'_-(0) = \lim_{\Delta x \rightarrow 0^-} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0^-} \frac{(\Delta x)^2 \sin \frac{1}{\Delta x}}{\Delta x} =$$

$$= \lim_{\Delta x \rightarrow 0^-} \Delta x \sin \frac{1}{\Delta x} = 0$$

$$f'_+(0) = \lim_{\Delta x \rightarrow 0^+} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \rightarrow 0^+} \Delta x \sin \frac{1}{\Delta x} = 0$$

$\Rightarrow f'_-(0) = f'_+(0) \Rightarrow f'(0)$ ima smisla i
 znaci $\boxed{f'(0) = 0}$

2201) Naci izvod slijedece f.c.je: $f(x) = 3 \frac{\sin ax}{\cos bx} + \frac{1}{3} \frac{\sin^3 ax}{\cos^3 bx}$

$$f(x) = f_1(x) + \frac{1}{3} f_2(x)$$

$$f'_1(x) = \left(3 \frac{\sin ax}{\cos bx} \right)' = 3 \frac{\sin ax}{\cos bx} \ln 3 \cdot \left(\frac{\sin ax}{\cos bx} \right)' =$$

$$= 3 \frac{\sin ax}{\cos bx} \cdot \ln 3 \cdot \frac{a \cdot \cos ax \cdot \cos bx + \sin ax \cdot b \cdot \sin bx}{\cos^2 bx} =$$

$$f'_2(x) = \left(\frac{\sin^3 ax}{\cos^3 bx} \right)' = \frac{3 \sin^2 ax \cdot a \cdot \cos ax \cdot \cos^3 bx + \sin^3 ax \cdot 3 \cos^2 bx \cdot b \cdot \sin bx}{\cos^6 bx} =$$

$$= 3 \cdot \frac{a \sin^2 ax \cdot \cos ax \cdot \cos^3 bx + b \cos^2 bx \cdot \sin^3 ax \cdot \sin bx}{\cos^5 bx}$$

$$\underline{\underline{f'(x) = f'_1(x) + \frac{1}{3} f'_2(x) = \left[3 \frac{\sin ax}{\cos bx} \ln 3 + \frac{\sin^2 ax}{\cos^2 bx} \right] \frac{a \cos ax \cos bx + b \sin ax \sin bx}{\cos^2 bx}}}$$

Naci izvod implicitno zadane f. je:

$$\begin{cases} x = x(t) \\ y = y(t) \end{cases} \Rightarrow y'_x = \frac{y'_t}{x'_t}$$

1) $x = e^t \sin t$

$y = e^t \cos t$

$$y'_x = \frac{y'_t}{x'_t} = \frac{\cancel{e^t} \cdot \cos t - \cancel{e^t} \sin t}{\cancel{e^t} \sin t + \cancel{e^t} \cos t} = \frac{\cos t - \sin t}{\cos t + \sin t}$$

2) $x = \frac{1+t^3}{t^2-1}$, $y = \frac{t}{t^2-1}$

$$y'_x = \frac{y'_t}{x'_t} = \frac{\frac{t^2-1-t(2t)}{(t^2-1)^2}}{\frac{2t^2(t^2-1) - (1+t^3) \cdot 2t}{(t^2-1)^2}} = \frac{t^2 - 2t^3 - 1}{3t^4 - 3t^2 - 2t - 2t^4} =$$

$$= - \frac{(t^2+1)}{t^4 - 3t^2 - 2t}$$

3) Pokazati da $y = f(x)$ zadan parametarski $x = 2t + 3t^3$;

$y = t^2 + 2t^3$ zadovoljava diferencijalnu jednačinu

$$2y^3 + y^2 - y = 0$$

$$y'_x = \frac{y'_t}{x'_t} = \frac{2t + 6t^2}{2 + 9t^2} = t$$

$$2t^3 + t^2 - t^2 - 2t^3 = 0 \quad \text{Ⓣ}$$

$$0 = 0$$